Subharmonic response of a single-degree-of-freedom nonlinear vibro-impact system to a narrow-band random excitation

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The subharmonic response of single-degree-of-freedom nonlinear vibro-impact oscillator with a one-sided barrier to narrow-band random excitation is investigated. The narrow-band random excitation used here is a filtered Gaussian white noise. The analysis is based on a special Zhuravlev transformation, which reduces the system to one without impacts, or velocity jumps, thereby permitting the applications of asymptotic averaging over the "fast" variables. The averaged stochastic equations are solved exactly by the method of moments for the mean-square response amplitude for the case of linear system with zero offset. A perturbation-based moment closure scheme is proposed and the formula of the mean-square amplitude is obtained approximately for the case of linear system with nonzero offset. The perturbation-based moment closure scheme is used once again to obtain the algebra equation of the mean-square amplitude of the response for the case of nonlinear system. The effects of damping, detuning, nonlinear intensity, bandwidth, and magnitudes of random excitations are analyzed. The theoretical analyses are verified by numerical results. Theoretical analyses and numerical simulations show that the peak amplitudes may be strongly reduced at large detunings or large nonlinear intensity.

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I. INTRODUCTION

Many nonsmooth factors arise very naturally in engineering applications, such as impacts, collisions, dry frictions, and so on. Most of the previous literatures fastened on the study of either nonsmooth nonlinear deterministic systems or nonsmooth linear stochastic systems. For deterministic nonsmooth systems, the dynamics of a periodically forced impact system was investigated in Ref. [1] and the bifurcations and chaos of two-degree-of-freedom linear vibro-impact systems were explored by the Pioncaré map in Refs. [2–6]. In recent years, the particular bifurcations unique to nonsmooth systems have been examined extensively [7–12]. In practice, engineering structures are often subjected to time-dependent loadings of stochastic nature, such as the natural phenomena due to wind gusts, earthquakes, ocean waves, and random disturbance or noise which always exists in a physical system. The influence of random excitation on the dynamical behavior of an impact dynamical system has caught the attention of many researchers. Some analysis methods, e.g., linearization method [13], quasistatic approach method [14,15], Markov processes method [16,17], stochastic averaging method [18,19], variable transformation method [20,21], energy balance method [22], mean impact Poincaré map method [23], and numerical simulation method [24], have been developed. In Ref. [25], the authors tried to review and summarize the existing methods, results, and literature available for solving problem of stochastic vibro-impact systems. However, most of researches are focused on responses of linear (here, "linear" means that the differential equation of motion between impacts is linear) impact oscillator under wide-band random excitations and few are focused on the responses of linear [26] especially of nonlinear impact oscillator under narrow-band random excitation. In Ref. [26], Dimentberg *et al.* discussed the subharmonic response of a linear impact system with a rigid one-sided barrier under a special kind of narrow-band random excitation—sinusoidal force with disorder or random-phase modulation—in detail. However, as pointed by Dimentberg *et al.*, for certain applications, the model of a filtered Gaussian white noise may be more appropriate for the basic narrow-band random excitation rather than one used in [26]. In this paper, the subharmonic response of single-degree-of-freedom nonlinear vibro-impact oscillator with a one-sided barrier, which is slightly offset from the system's equilibrium position, under the general narrow-band random excitation—a filtered Gaussian white noise—is investigated. The impact considered here is an instantaneous impact with restitution factor e. The paper is organized as follows. In Sec. II, the Zhuravlev transformation and stochastic averaging method are used to obtain the mean-square amplitude of the response. In Sec. III, the directly numerical simulations verify the analytical result. Conclusions are presented in Sec. IV.

II. SYSTEM DESCRIPTION AND THEORETICAL ANALYSES

Considering the single-degree-of-freedom nonlinear vibro-impact oscillator to random excitations

$$\begin{cases} \ddot{y} + 2\beta\dot{y} + \Omega^2 y + \alpha y^3 = h\xi(t), & y < \Delta \\ \dot{y}_+ = -e\dot{y}_-, & y = \Delta, \end{cases} \tag{1}$$

where dot indicates differentiation with respect to time t, β and Ω are damping coefficient and natural frequency, respectively, α represents the intensity of the nonlinear term, Δ

represents the distance from the system's static equilibrium position to the single rigid barrier, $0<e\leq 1$ is the restitution factor to be a known parameter of impact losses, whereas subscripts "minus" and "plus" refer to values of response velocity just before and after the instantaneous impact. Thus $\dot{y}_+$ and $\dot{y}_-$ are actually rebound and impact velocities of the mass, respectively. They have the same magnitude whenever $e=1$, therefore, this special case is that of elastic impacts, whereas in case $e<1$, some impact losses are observed. h denotes the intensity of the random excitation and $\xi(t)$ is chosen to be a zero-mean Gaussian narrow-band random excitation. It could be obtained by filtering a white noise through a linear filter, that is

$$\ddot{\xi}+\gamma\dot{\xi}+\Omega_1^2\xi=\sqrt{\gamma}\Omega_1 W, \tag{2}$$

where Ω_1 is the center frequency of $\xi(t)$, γ is the bandwidth of the filter, and $W(t)$ is a unit white noise with the autocorrelation function $R_W(\tau)=2\pi\delta(\tau)$, and δ is the Dirac delta function. The subharmonic response of system (1) is discussed in detail by Dimentberg *et al.* [26] for the case of linear system ($\alpha=0$) and $\xi(t)$ is a sinusoidal force with disorder as the following:

$$\xi(t)=\sin\varphi(t),\quad \dot{\varphi}(t)=\Omega_1+\gamma W(t). \tag{3}$$

However, as pointed by Dimentberg *et al.*, for certain applications, the model of a filtered Gaussian white noise governed by Eq. (2) may be more appropriate for the basic narrow-band random excitation rather than one used in [26]. From Eq. (2), $\xi(t)$ may be rewritten as [27]

$$\xi(t)=\xi_1(t)\sin\Omega_1 t+\xi_2(t)\cos\Omega_1 t, \tag{4}$$

where $\xi_1(t)$ and $\xi_2(t)$ are slowly varying random functions of time. In fact, substituting Eq. (4) into Eq. (2) and performing deterministic and stochastic averagings of the equations describing the modulations of $\xi_1(t)$ and $\xi_2(t)$ with time, one obtains

$$\dot{\xi}_1+\frac{\gamma}{2}\xi_1=\sqrt{\frac{\gamma}{2}}W_1,\quad \dot{\xi}_2+\frac{\gamma}{2}\xi_2=\sqrt{\frac{\gamma}{2}}W_2. \tag{5}$$

The unit white-noise components W_1 and W_2 are independent. The autocorrelation functions of $\xi_1(t)$ and $\xi_2(t)$ are

$$R_{\xi_1}(\tau)=R_{\xi_2}(\tau)=\pi e^{-\gamma|\tau|/2}.$$

The correlation time of $\xi_1(t)$ and $\xi_2(t)$ is $O(\frac{1}{\gamma})$ and this means that for sufficiently small values of γ, $\xi_1(t)$ and $\xi_2(t)$ are slowly varying random functions of time. From Eq. (4), $\xi(t)$ may be rewritten as

$$\xi(t)=\sqrt{\xi_1^2+\xi_2^2}\sin[\Omega_1 t+\varphi(t)],\quad \varphi(t)=\arctan\frac{\xi_2}{\xi_1}. \tag{6}$$

Obviously, $\varphi(t)$ is also a slowly varying random function of time.

Following Zhuravlev [20], the nonsmooth transformation of state variables is introduced as follows:

$$y=|x|+\Delta,\quad \dot{y}=\dot{x}\,\mathrm{sgn}\,x, \tag{7}$$

where $\mathrm{sgn}\,x$ is the signum function such that $\mathrm{sgn}\,x=1$ for $x>0$ and $\mathrm{sgn}\,x=-1$ for $x<0$. Obviously, this transformation makes the transformed velocity $\dot{x}$ continuous at the impact instants (i.e., $x=0$) in the special case of elastic impact (i.e., $e=1$), thereby reducing the problem to one without velocity jumps. However, this is not the case with a general vibro-impact system with impact losses. The jump of the transformed velocity $\dot{x}$ becomes proportional to $1-e$ instead of $1+e$ for the jump of original velocity $\dot{y}$. This jump may be included in the transformed differential equation of motion by using the Dirac delta function $\delta(x)$. Since $x(t_*)=0$ at the impact instant t_* and $\delta(t-t_*)=|\dot{x}|\delta(x)$, the impulsive term can be obtained as

$$(\dot{x}_+-\dot{x}_-)\delta(t-t_*)=(1-e)\dot{x}|\dot{x}|\delta(x).$$

The transformed equation of motion can be written by substituting Eqs. (6) and (7) into Eq. (1) as

$$\ddot{x}+\Omega^2x=-2\beta\dot{x}-\Delta\Omega^2\,\mathrm{sgn}\,x-(1-e)\dot{x}|\dot{x}|\delta(x)-\alpha(|x|+\Delta)^3\mathrm{sgn}\,x+h\sqrt{\xi_1^2+\xi_2^2}\,\mathrm{sgn}\,x\sin[\Omega_1 t+\varphi(t)]. \tag{8}$$

Thus, the original impact system (1) is reduced to the "common" vibration system (8) without impact. The term $(1-e)\dot{x}|\dot{x}|\delta(x)$ on the right-hand side of Eq. (8) describes the impact losses of system, which can be regarded as an impulsive damping term. The transformed equation (8) permits rigorous analytical study by the asymptotic method of averaging over the period as long as coefficients α,β,Δ,h, and $1-e$ are all small and proportional to a small parameter. Moreover, only subharmonic resonant responses will be considered, i.e., frequency Ω_1 of the random excitation is near the subharmonic resonant frequency $2n\Omega$, $\Omega_1\approx 2n\Omega$, where n is an arbitrary positive integer. The detuning parameter μ is defined according to $\mu=\Omega_1-2n\Omega$; μ is assumed to be small and proportional to a small parameter. Then the right-hand side of Eq. (8) is small and one has $\ddot{x}+\Omega^2x\approx 0$. The general solution of equation $\ddot{x}+\Omega^2x=0$ is

$$x=C_1\sin(\Omega t+C_2),\quad \dot{x}=\Omega C_1\cos(\Omega t+C_2),$$

where C_1 and C_2 are arbitrary constants. Then, as usually do in stochastic averaging progress, the response of Eq. (8) can be approximately represented as

$$x=A(t)\sin\Phi(t),\quad \dot{x}=\Omega A(t)\cos\Phi(t). \tag{9}$$

By introducing a new slowly varying phase shift $\theta(t)=\Omega_1 t+\varphi(t)-2n\Phi(t)$, Eq. (8) can be transformed to the following pair of first-order equations:

$$\begin{cases} \dot{A} = \dfrac{\cos\Phi}{\Omega}[-2\beta\Omega A\cos\Phi - \Delta\Omega^2\,\mathrm{sgn}(\sin\Phi) - (1-e)\Omega^2 A\cos\Phi|A\cos\varphi|\delta(A\sin\Phi) \\ \quad -\alpha(|A\sin\Phi|+\Delta)^3\mathrm{sgn}(\sin\Phi) + h\sqrt{\xi_1^2+\xi_2^2}\,\mathrm{sgn}(\sin\Phi)\sin(\theta+2n\Phi)] \\ \dot{\theta} = \mu + \dfrac{2n\sin\Phi}{\Omega A}\Bigg[-2\beta\Omega A\cos\Phi - \Delta\Omega^2\,\mathrm{sgn}(\sin\Phi) - (1-e)\Omega^2 A\cos\Phi|A\cos\varphi|\delta(A\sin\Phi) \\ \quad -\alpha(|A\sin\Phi|+\Delta)^3\mathrm{sgn}(\sin\Phi) + h\sqrt{\xi_1^2+\xi_2^2}\,\mathrm{sgn}(\sin\Phi)\sin(\theta+2n\Phi) + \sqrt{\dfrac{\gamma}{2}}\dfrac{W_2\xi_1 - W_1\xi_2}{\xi_1^2+\xi_2^2}.\Bigg] \end{cases} \tag{10}$$

Under the foregoing assumption that damping, intensity of nonlinear term, impact losses, and excitation terms are small, the right-hand sides of both Eqs. (10) are proportional to a small parameter, the derivatives $\dot{A}$ and $\dot{\theta}$ are both small, then A,θ are all slowly varying random processes with respect to time t. According to the definition of $\theta(t)$, one has

$$\Phi(t) = \frac{1}{2n}[\Omega_1 t + \varphi(t) - \theta(t)], \quad \dot{\Phi} = \frac{1}{2n}(\Omega_1 + \dot{\varphi} - \dot{\theta}),$$

where $\varphi(t)$ and $\theta(t)$ are both slowly varying random functions of time and the derivatives $\dot{\varphi}$ and $\dot{\theta}$ are both small, while $\dot{\Phi}\approx\frac{\Omega_1}{2n}$ is not small and then Φ is a fast varying random process. By averaging over the fast state variables Φ, W_1 and W_2, the following shortened equations can be obtained:

$$\begin{cases} \dot{A} = -\beta_1 A + q\zeta\cos\theta \\ \dot{\theta} = \left(\mu - \dfrac{3n\alpha\Delta^2}{\Omega}\right) - \dfrac{q}{A}\zeta\sin\theta - \dfrac{\delta}{A} - \dfrac{n\alpha}{\Omega}\left(\dfrac{8}{\pi}\Delta A + \dfrac{3}{4}A^2\right) \\ \beta_1 = \beta + \dfrac{1-e}{\pi}\Omega, q = \dfrac{4nh}{(4n^2-1)\pi\Omega}, \delta = \dfrac{4n\Omega\Delta}{\pi}(1-\alpha\Delta^2), \zeta = \sqrt{\xi_1^2+\xi_2^2}. \end{cases} \tag{11}$$

Equation (11) shows that the difference between elastic impact ($e=1$) and inelastic impact ($e<1$) is that inelastic impact increases the damping of the system from β to $\beta_1=\beta+\frac{1-e}{\pi}\Omega$. Introducing another new pair of state variables

$$u = A\cos\theta, \quad v = A\sin\theta, \tag{12}$$

Eq. (11) can be transformed to

$$\begin{cases} \dot{u} = -\beta_1 u - \left(\mu - \dfrac{3n\alpha\Delta^2}{\Omega}\right)v + \dfrac{\delta v}{\sqrt{u^2+v^2}} + \dfrac{n\alpha v}{\Omega}\left[\dfrac{8}{\pi}\Delta\sqrt{u^2+v^2} + \dfrac{3}{4}(u^2+v^2)\right] + q\zeta, \\ \dot{v} = -\beta_1 v + \left(\mu - \dfrac{3n\alpha\Delta^2}{\Omega}\right)u - \dfrac{\delta u}{\sqrt{u^2+v^2}} - \dfrac{n\alpha u}{\Omega}\left[\dfrac{8}{\pi}\Delta\sqrt{u^2+v^2} + \dfrac{3}{4}(u^2+v^2)\right]. \end{cases} \tag{13}$$

It should be noted that an exact analytical study to system (13) seems impossible due to nonlinear nature. Thus, approximate solutions of the second-order moments of the subharmonic response are proposed. Equation (13) seems to be linear if $\alpha=0,\ \Delta=0(\delta=0)$ and thus is amenable to an exact analysis by the method of moments [28]. For the case of linear system with zero offset ($\alpha=0,\ \Delta=0$), Eq. (13) can be written as

$$\dot{u} = -\beta_1 u - \mu v + q\zeta, \quad \dot{v} = -\beta_1 v + \mu u. \tag{14}$$

While the response moments of any order can be predicted easily from Eq. (14), only the mean-square amplitude $EA^2=E(u^2+v^2)$ will be considered here, where E denotes the mathematics expectation. For steady-state response, one has

$$\frac{dEu^2}{dt} = \frac{dEv^2}{dt} = \frac{dEuv}{dt} = \frac{dEu\zeta}{dt} = \frac{dEv\zeta}{dt} = 0. \tag{15}$$

From Eqs. (5), (14), and (15), applying the expectation operator and equating the time derivative to zero for a steady-state solution yield

$$\begin{cases} -\beta_1 Eu^2 - \mu Euv + qEu\zeta = 0, \mu Euv - \beta_1 Ev^2 = 0 \\ \mu Eu^2 - 2\beta_1 Euv - \mu Ev^2 + qEv\zeta = 0 \\ \left(\beta_1 + \dfrac{\gamma}{2}\right)Eu\zeta + \mu Ev\zeta = qE\zeta^2, \mu Eu\zeta - \left(\beta_1 + \dfrac{\gamma}{2}\right)Ev\zeta = 0. \end{cases} \tag{16}$$

Form Eq. (5), one has $E\zeta^2=E(\xi_1^2+\xi_2^2)=1$. Substituting this expression into Eq. (16), one can obtain the solution of Eq. (16) as

$$\begin{cases} Eu^2=\dfrac{q^2\left(\beta_1^3+\dfrac{\beta_1^2\gamma}{2}+\dfrac{\mu^2\gamma}{4}\right)}{\beta_1(\beta_1^2+\mu^2)\left[\left(\alpha+\dfrac{\gamma}{2}\right)^2+\mu^2\right]}, Euv=\dfrac{q^2\mu\left(\beta_1+\dfrac{\gamma}{4}\right)}{(\beta_1^2+\mu^2)\left[\left(\beta_1+\dfrac{\gamma}{2}\right)^2+\mu^2\right]}, \\ Ev^2=\dfrac{q^2\mu^2\left(\alpha+\dfrac{\gamma}{4}\right)}{\beta_1(\beta_1^2+\mu^2)\left[\left(\beta_1+\dfrac{\gamma}{2}\right)^2+\mu^2\right]}, Eu\zeta=\dfrac{q\left(\beta_1+\dfrac{\gamma}{2}\right)}{\left(\beta_1+\dfrac{\gamma}{2}\right)^2+\mu^2}, Ev\zeta=\dfrac{q\mu}{\left(\beta_1+\dfrac{\gamma}{2}\right)^2+\mu^2}. \end{cases} \tag{17}$$

Then mean-square amplitude can be obtained from Eq. (17) as

$$EA^2=Eu^2+Ev^2=\frac{q^2\left(\beta_1+\dfrac{\gamma}{2}\right)}{\beta_1\left[\left(\beta_1+\dfrac{\gamma}{2}\right)^2+\mu^2\right]}. \tag{18}$$

In can be seen from formula (18) that the mean-square amplitude-frequency response curves are symmetric with respect to the corresponding resonant frequency of any given order n, i.e., same for positive and negative detunings.

Next, we discuss the mean-square amplitude of system (13) for the case of linear system ($\alpha=0$), i.e., system (1) is linear, with nonzero offset $\Delta\neq 0$. Equation (13) can be written as

$$\begin{cases} \dot{u}=-\beta_1 u-\mu v+\dfrac{\delta v}{\sqrt{u^2+v^2}}+q\zeta \\ \dot{v}=-\beta_1 v+\mu u-\dfrac{\delta u}{\sqrt{u^2+v^2}}. \end{cases} \tag{19}$$

As long as the Eq. (19) is nonlinear for nonzero Δ, this requires use of some closure scheme. Denoting $A_*^2=EA^2=E(u^2+v^2)$ and substituting all $\sqrt{u^2+v^2}$ terms in the right-hand side of Eq. (19) by A_*, Eq. (19) can be transformed to the following linear equation:

$$\dot{u}=-\beta_1 u-\left(\mu-\frac{\delta}{A_*}\right)v+q\zeta,\quad \dot{v}=-\beta_1 v+\left(\mu-\frac{\delta}{A_*}\right)u. \tag{20}$$

Equation (20) is in the same form with Eq. (14), then from formula (18), one obtains the equation of A_* as

$$A_*^2=\frac{q^2\left(\beta_1+\dfrac{\gamma}{2}\right)}{\beta_1\left[\left(\beta_1+\dfrac{\gamma}{2}\right)^2+\left(\mu-\dfrac{\delta}{A_*}\right)^2\right]}, \tag{21}$$

Equation (21) has the following solution:

$$A_*=\frac{\beta_1\mu\delta\pm\sqrt{\beta_1^2\mu^2\delta^2+\beta_1\left[\left(\beta_1+\dfrac{\gamma}{2}\right)^2+\mu^2\right]\left[q^2\left(\beta_1+\dfrac{\gamma}{2}\right)-\beta_1\delta^2\right]}}{\beta_1\left[\left(\beta_1+\dfrac{\gamma}{2}\right)^2+\mu^2\right]}. \tag{22}$$

It should be noted that the foregoing method to obtain the approximate solution of Eq. (19) has been presented by Dimentberg *et al.* [26] and will be used again for the case of nonlinear system ($\alpha\neq 0$) in the following discussion. It can clearly be seen that with $\Delta\to 0$ ($\delta\to 0$), the squared quantity A_*^2 approaches the "exact" mean-square amplitude EA^2 as governed by Eq. (18).

Now we discuss the mean-square amplitude of system (13) for the case of nonlinear system ($\alpha\neq 0$). Substituting all $\sqrt{u^2+v^2}$ terms in the right-hand side of Eq. (13) by A_*, Eq. (13) can be transformed to the following linear equation:

$$\begin{cases} \dot{u} = -\beta_1 u - \left[\mu - \dfrac{3n\alpha\Delta^2}{\Omega} - \dfrac{\delta}{A_*} - \dfrac{n\alpha}{\Omega}\left(\dfrac{8}{\pi}\Delta A_* + \dfrac{3}{4}A_*^2\right)\right]v + q\zeta \\ \dot{v} = -\beta_1 v + \left[\mu - \dfrac{3n\alpha\Delta^2}{\Omega} - \dfrac{\delta}{A_*} - \dfrac{n\alpha}{\Omega}\left(\dfrac{8}{\pi}\Delta A_* + \dfrac{3}{4}A_*^2\right)\right]u. \end{cases} \tag{23}$$

Equation (23) is also in the same form with Eq. (14), then from formula (18), one obtains the equation of A_* as

$$A_*^2 = \frac{q^2\left(\beta_1 + \dfrac{\gamma}{2}\right)}{\beta_1\left[\left(\beta_1 + \dfrac{\gamma}{2}\right)^2 + \left(\mu - \dfrac{3n\alpha\Delta^2}{\Omega} - \dfrac{\delta}{A_*} - \dfrac{8n\alpha\Delta}{\pi\Omega}A_* - \dfrac{3n\alpha}{4\Omega}A_*^2\right)^2\right]}. \tag{24}$$

Equation (24) can be solved numerically for given parameters of the system.

III. NUMERICAL SIMULATION

In this section, the analytical results will be shown and compared to the directly numerical results. All the directly numerical simulations by using Monte Carlo method are based on the original system dominated by Eq. (1), which gives powerful validation with analytical results. For the method of numerical simulation, readers can refer to Zhu [28] and Shinozuka [29]. The unit white noise $W(t)$ in Eq. (2) has the power spectrum of a constant over the spectrum range $0<\omega<\infty$ such that $W(t)$ has infinite energy and is physical unrealized. However, for numerical simulation in this paper, the power spectrum of $W(t)$ is taken as

$$S(\omega) = \begin{cases} \dfrac{1}{2\pi}, & 0<\omega\le 2\Omega \\ 0, & \omega > 2\Omega. \end{cases}$$

For numerical simulation, it is more convenient to use the pseudorandom signal given by [28]

$$W(t) = \sqrt{\frac{2\Omega}{N\pi}}\sum_{k=1}^{N}\cos\left[\frac{\Omega}{N}(2k-1)t + \varphi_k\right],$$

where φ_k's are independent and uniformly distributed in $(0,2\pi]$ and N is a larger integer number. A realization of $W(t)$ is shown in Fig. 1 and the sample mean is -0.0264.

Monte Carlo simulations are focused on the first-order subharmonics ($n=1$, $\Omega_1 \approx 2\Omega$), although the higher-order subharmonics ($\Omega_1 \approx 2n\Omega$, $n=2,3,4,\ldots$) simulations should be of the same importance. The governing equations (1) and (2) are numerically integrated by the fourth-order Runge-Kutta algorithm between impacts, which is valid until the first encounter with the barriers, that is until the equality $y=\Delta$ is satisfied. The impact condition $\dot{y}_+=-e\dot{y}_-$ is then imposed using the numerical solution $\dot{y}_-$. This results in the rebound velocity $\dot{y}_+$, thereby providing the initial values for the next step numerical calculation. In fact, the numerical solution $y(t)$ will be approximately equal to Δ $(y(t)\approx\Delta)$, then some interpolation techniques should be used to determine the exact impact time $t=t_*$ such that $y(t_*)=\Delta$. In this paper, the linear interpolation method is used to determine the exact impact time $t=t_*$ and impact velocity $\dot{y}_-=\dot{y}(t_*-0)$. The following linear vibro-impact oscillator to deterministic harmonic excitations is used to verify the accuracy of the Runge-Kutta algorithm and interpolation technique

$$\begin{cases} \ddot{y} + y = h\cos(\Omega_1 t + \psi), & y<\Delta \\ \dot{y}_+ = -e\dot{y}_-, & y=\Delta, \end{cases} \tag{25}$$

It is easy to obtain the response of system (25). Such response may exist with ratio $\frac{1}{l}$ of its period to the period of excitation with integer l and satisfies the following conditions:

$$y(0)=\Delta,\quad \dot{y}(0)=\dot{y}_+,\quad y\left(\frac{2\pi l}{\Omega_1}\right)=\Delta,\quad \dot{y}\left(\frac{2\pi l}{\Omega_1}\right)=\dot{y}_-. \tag{26}$$

The steady-state response of system (26) can be obtained as

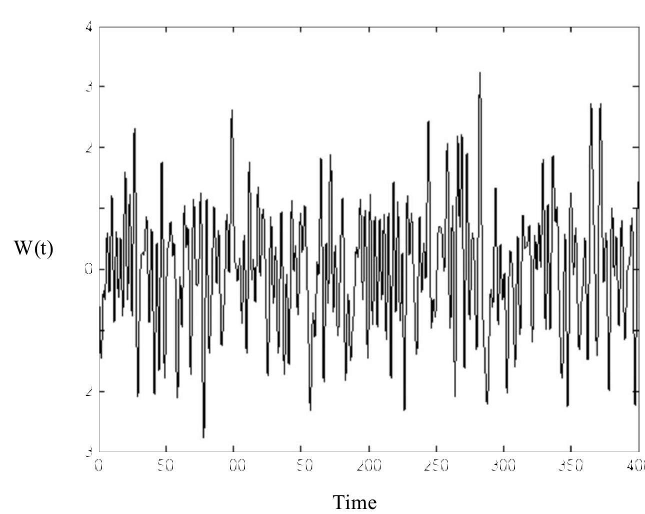

FIG. 1. A realization of $W(t)$.

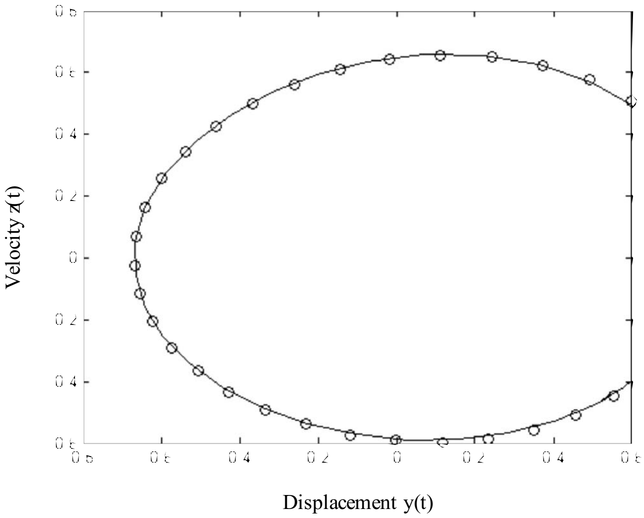

FIG. 2. Numerical results of Eq. (25): —theoretical solution; ○○○ numerical solution.

$$y(t) = y_0(t) = a_\omega \cos\left(t - \frac{\pi l}{\Omega_1}\right) + a_{\Omega_1} \cos(\Omega_1 t + \psi), \quad (27)$$

where

$$a_\omega = \frac{\Delta - a_{\Omega_1} \cos \psi}{\cos(\pi l/\Omega_1)}, \quad a_{\Omega_1} = \frac{h}{|1 - \Omega_1^2|}.$$

The impact velocity is

$$\dot{y}_- = \frac{[1 \pm \sqrt{1 - (1 - a_{\Omega_1}^2/\Delta^2)^2(1 + B^2)}]\Delta}{(1 + B^2)D}, \quad (28)$$

where

$$B = \frac{1}{\Omega_1}\frac{1 - e}{1 + e}\tan\frac{\pi l}{\Omega_1}, \quad D = -\frac{1}{2}(1 + e)\cot\frac{\pi l}{\Omega_1}. \quad (29)$$

The numerical results of system (25) are shown in Fig. 2 when $h=2$, $\Delta=0.6$, $\Omega_1=1.1$, $e=0.8$, where $z(t)=\dot{y}(t)$ denote the velocity of the mass. For comparison, the theoretical results given by Eq. (27) are also shown in Fig. 2. Figure 2 shows that the numerical results are in good agreement with the theoretical solutions.

In the following numerical simulation, the parameters in system (1) are chosen as follows:

$$h = 2.0, \quad \Delta = 0.1, \quad \Omega = 1, \quad n = 1.$$

The numerical results are shown in Figs. 3 and 4. There are no units of the axes in all figures since system (1) is a nondimensional system.

We first consider the effect of the damping coefficient β and center frequency Ω_1 on the response amplitude A_* of the system. The variations of the steady-state response A_* with Ω_1 are shown in Fig. 3(a) for the case of linear system as $\alpha=0$, $\gamma=0.1$, $e=0.9$, when $\beta=0.15$ and $\beta=0.1$. For comparison, the theoretical results given by Eq. (22) are also shown in Fig. 3(a). The mean-square response amplitude was calculated as $A_*^2=2\langle(\frac{\dot{x}}{\Omega})^2\rangle$ in numerical simulation, where an-

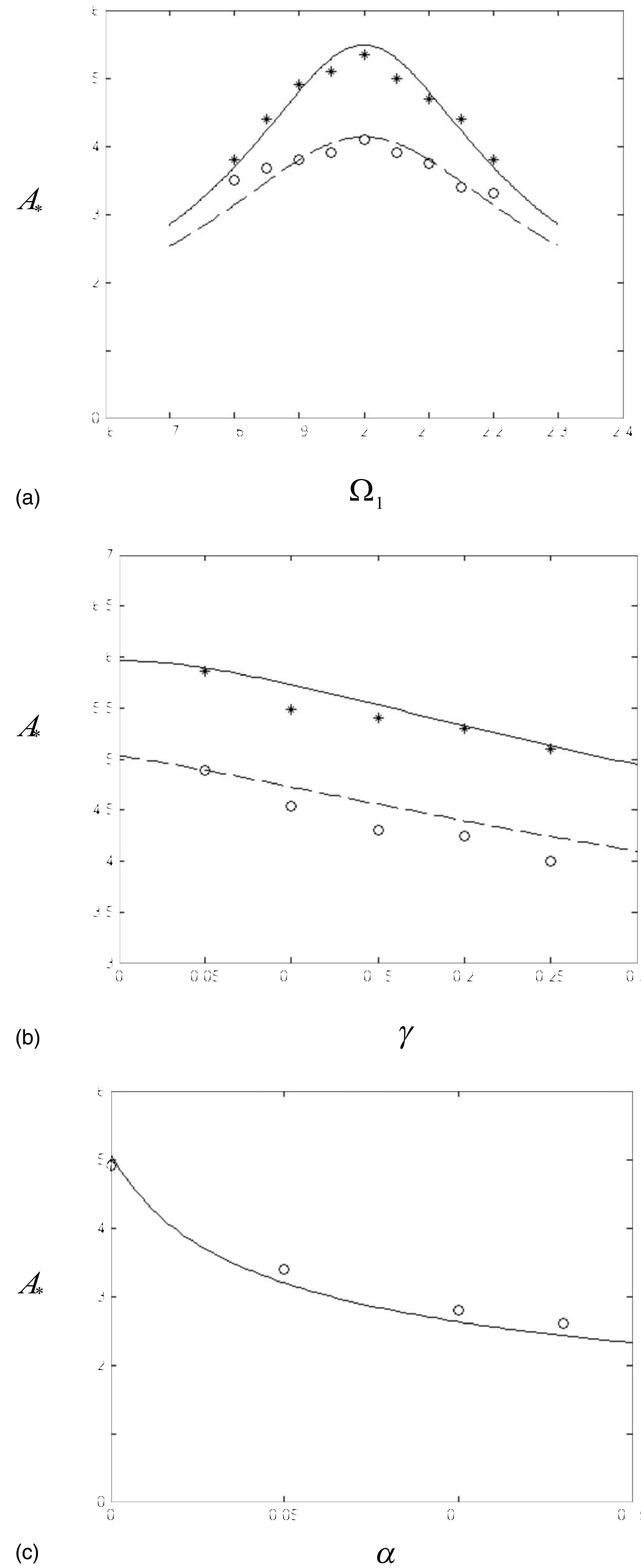

FIG. 3. Frequency response of system (1): —, ———: theoretical solution; ○○○, ***: numerical solution. (a) ($\alpha=0$, $\gamma=0.1$, $e=0.9$), —: $\beta=0.1$, ———: $\beta=0.15$, ***: ($\beta=0.1$, ○○○: $\beta=0.15$; (b) ($\alpha=0.0$, $\Omega_1=2.08$, $\beta=0.1$), —: $e=1.0$, ———: $e=0.9$, ***: $e=1.0$, ○○○: $e=0.9$; (c) ($\Omega_1=1.9$, $\beta=0.1$, $e=0.9$, $\gamma=0.1$).

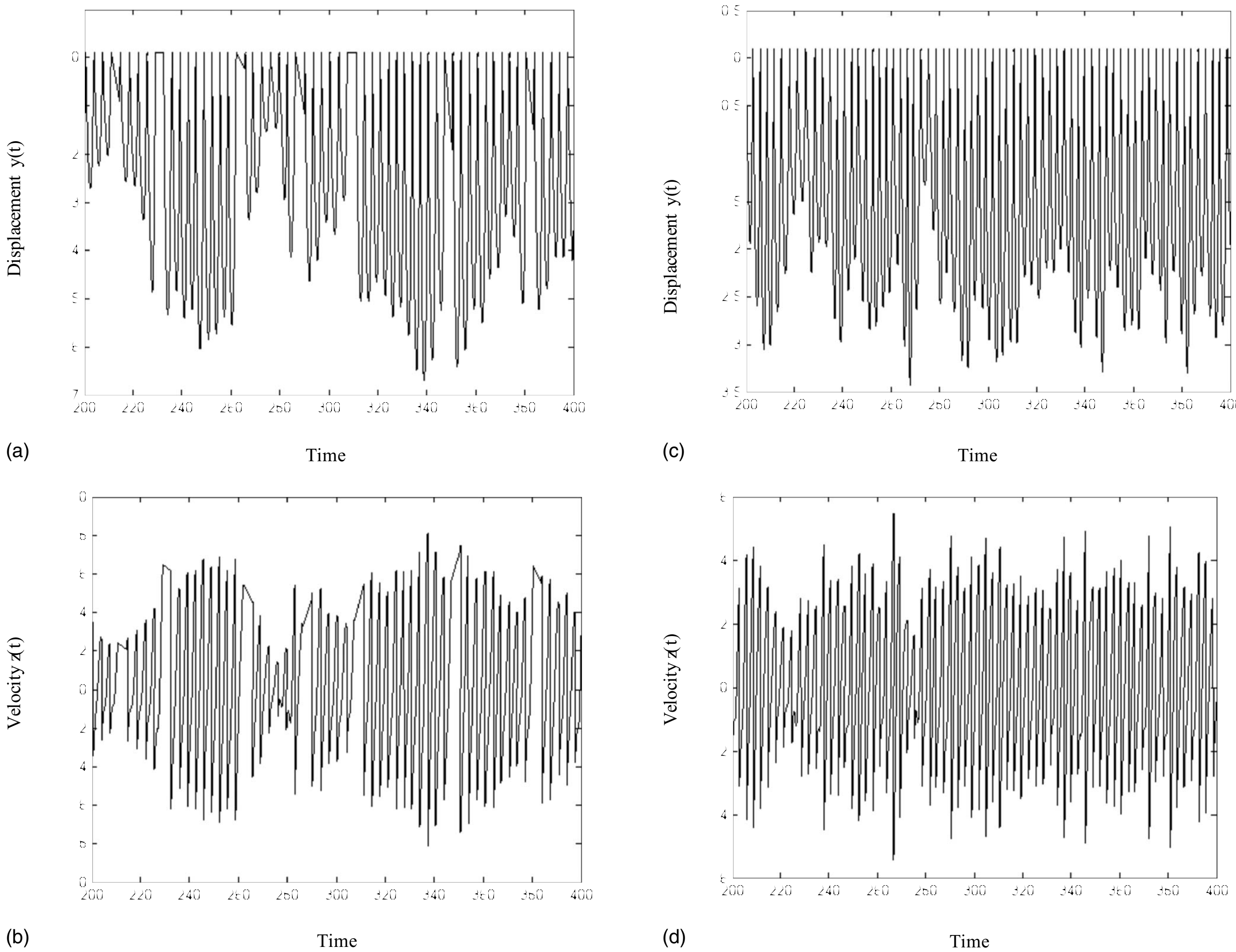

FIG. 4. Numerical results of Eq. (1). [(a) and (c)] Time history of $y(t)$; [(b) and (d)] Time history of $z(t)$. [(a) and (b)] (α=0, Ω_1=2.08, β=0.1, e=1.0, γ=0.1), [(c) and (d)] (α=0.1, Ω_1=2.08, β=0.1, e=1.0, γ=0.1).

gular brackets denote common time averaging for the response sample. Figure 3(a) shows that the response amplitude predicted by the averaging method is in good agreement with that obtained by numerical results. It can be seen from Fig. 3(a) that the response amplitude will decrease when the damping β increases, which is in accordance with the physical intuition. The peak response amplitude will become large when the frequency Ω_1 is near the resonant frequency $\Omega_1=2$ and will decrease strongly when Ω_1 departs from the resonant frequency. Comparing to the numerical solution, the accuracy of the analytical solution is seen to be reduced a little in the case of large detuning. This may be partly due to some inaccuracies of the averaging method at large detuning.

Next, we consider the effect of the restitution factor e and the bandwidth γ on the response amplitude A_* of the system. The variations of the steady-state response A_* with γ are shown in Fig. 3(b) for the case $\alpha=0.0$, $\Omega_1=2.08$, $\beta=0.1$, when $e=1.0$ and $e=0.9$. For comparison, the theoretical results given by Eq. (22) are also shown in Fig. 3(b). It can be seen from Fig. 3(b) that the response amplitude A_* will decrease when γ increases or e decrease. There are some impact losses when $e<1$ and the impact losses will increase when e decreases, leading effectively to a large viscous damping factor, which is in accordant with the foregoing theoretical analyses.

Now we consider the effect of the intensity of nonlinearity α on the response amplitude A_* of the system. The variations of the steady-state response A_* with α are shown in Fig. 3(c) for the case $\Omega_1=1.9$, $\beta=0.1$, $e=0.9$, $\gamma=0.1$. For comparison, the theoretical results given by Eq. (24) are also shown in Fig. 3(c). It can be seen from Fig. 3(c) that the response amplitude A_* will decrease strongly when α increases, therefore the nonlinearity should be considered in the analysis of impact system.

The response time history of system (1) is shown in Figs. 4(a) and 4(b) in the case $\Omega_1=2.08$, $\alpha=0$, $\beta=0.1$, $e=1.0$, $\gamma=0.1$. The time starts from zero in numerical simulation and the response sample that only starts from 200 is used such that the response is a stable steady one. The response time history of system (1) is shown in Figs. 4(c) and 4(d) in the case $\Omega_1=2.08$, $\alpha=0.1$, $\beta=0.1$, $e=1.0$, $\gamma=0.1$, for comparison to Figs. 4(a) and 4(b). All the parameters are the same in Figs. 4(a) and 4(b) and Figs. 4(c) and 4(d) except for α, while $\alpha=0$ in Figs. 4(a) and 4(b) and $\alpha=0.1$ in Figs.

4(c) and 4(d). It can be seen clearly from Figs. 4(a)–4(d) that the response amplitude will decrease strongly when α increases.

IV. CONCLUSIONS AND DISCUSSION

In this paper, the response of a nonlinear impact system under narrow-band random excitation—a filtered Gaussian white noise—is discussed. The steps of the approach presented in the paper may be summarized as follows. The filtered Gaussian white noise is rewritten in a sinusoidal force form for convenient of theoretical analyses first. Then the method of Zhuravlev transformation is used to translate the original impact system to a common vibration system without impact, which can be reduced to shortened equations by averaging the fast state variables. The reduced equations are solved by the method of moments and exactly analytical solution for the mean-square subharmonic response amplitude is obtained for the case of linear system with zero offset, while an approximate perturbational closure scheme is used for the general case. Comparison to Monte Carlo simulation results demonstrated high accuracy of the analytical solution. The basic conclusion of the analysis may be that the peak response amplitude will become large when the frequency Ω_1 is near the resonant frequency $\Omega_1=2\Omega$ and will decrease strongly when Ω_1 departs from the resonant frequency or α increase. The peak response amplitude will also decrease when γ and β increase or e decreases.

Small parameters are often required in many averaging and perturbation methods. However, it is difficult to specify what they mean for "small" in generally. In this paper, the theoretical derivations are valid for "small values" of some of the parameters. The values of these small parameters are taken as $O(1)$ in numerical simulation.

The approach presented in this paper is appropriate for the narrow-band random noise in more general case as governed by Eq. (4), where $\xi_1(t)$ and $\xi_2(t)$ are slowly varying random function and are not only defined by Eq. (5). However, Eq. (2) is a well-known model and has been widely used, therefore is presented in this paper. For a comprehensive survey of narrow-band random noise, the reader is referred to Stratonovich [14] and Zhu [28].

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